ABSTRACT ALGEBRA-II

SMMA51

Vector Space

Definition:

A non empty set V is said to be a vector space over a field F if

1) V is an abelian group under an operation called addition which we defined by +.
2) For every $\alpha \varepsilon F$ and $v \varepsilon V$, there is defined an element αv in V subject to the following conditions.

   a) $\alpha(u+v)=\alpha u+\alpha v$ for all $u,v\varepsilon V$ and $\alpha \varepsilon F$.

   b) $(\alpha+\beta)u=\alpha u+\beta u$ for all $u\varepsilon v$ and $\alpha,\beta \varepsilon F$.

   c) $\alpha(\beta u)=(\alpha\beta)u$ for all $u\varepsilon V$ and $\alpha,\beta\varepsilon F$.

   d) $1u=u$ for all $u\varepsilon V$.

Remarks:

1. The elements of F are called scalars and the elements of V are called vectors.
2. The rule which associates with each scalar $\alpha\varepsilon F$ and a vector $v\varepsilon V$, a vector αv is called the scalar multiplication. Thus a scalar multiplication gives rise to a function from $F\times v\rightarrow V$ defined by $(\alpha,v)\rightarrow\alpha v$.

Examples:

1) Let F be any field

   Let $F^n=\{(x_1,x_2.....x_n)/x_i \varepsilon F\}$

   In f^n we define addition and scalar multiplication as in example.

Then F^n is a vector space over F and denote this vector space by $V_n(F)$.

2) Let F be a field . Then F[x] the set of all polynomials over F is a vector space over F under the addition of polynomials and scalar multiplication defined by

$\alpha(a_0+a_1x+\dots\dots+a_nx^n)= \alpha a_0+\alpha a_1x+\dots\dots+\alpha a_nx^n$

3) Let V be the set of all funtions from R to R

Let f,gεV we define(f+g)(x)=f(x)+g(x) and

$(\alpha f)(x)=\alpha[f(x)]$

V is a vector space over R.

Theorem:5.1

Let V be a vector space over a field F.Then

1.$\alpha 0=0$ for all $\alpha\varepsilon F$.

2.$0v=0$ for all $v\varepsilon V$

3.$(-\alpha)V=\alpha(-V)=-(\alpha V)$ for all $\alpha\varepsilon F$ and $v\varepsilon V$.

4. $\alpha v=0 \Rightarrow \alpha=0$ or $v=0$

Proof:

1) $\alpha 0=\alpha(0+0)= \alpha 0+\alpha 0$

Hence $\alpha 0=0$

2) $0v=(0+0)v=0v+0v$

Hence $0v=0$

3)$0=[\alpha+(-\alpha)]v$

$=\alpha v+(-\alpha)v$

Hence $(-\alpha)v=-(\alpha v)$

Similarly $\alpha(-v)=-(\alpha v)$

Hence $(-\alpha)v=\alpha(-v)=-(\alpha v)$

4) Let $uv=0$ if $\alpha=0$ there is nothing to prove.

Let $\alpha\neq 0$ Then $\alpha^{-1}\varepsilon F$.

Now $v=1v=(\alpha^{-1}\alpha)v=\alpha^{-1}(\alpha v)$

$=\alpha^{-1}0=0.$

Subspaces

Definition:

Let V be a vector space over a field F. A nonempty subset W of V is called a subspace of V if W itself is a vector space over F under the operation of V

Theorem:5.2

Let V be a vector space over F.A nonempty subset W of V is a subspaces of V iff W is closed with respect to vector addition and scalar multiplication in V.

Soln:

Let W be a subspace of V.

W itself is a vector space and W is closed with respect to vector addition and scalar multiplication.

Conversely, let W be a non-empty subset of V such that $u,v \in W$

$\Rightarrow u + v \in W$

Scalar multiplication

$u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$.

We prove that W is a subspace of V.

W is nonempty there exists an element $u \in W$

Identy :

$0u = 0 \in W$

Inverse:

$V \in W \Rightarrow (-1) V$

$\Rightarrow -v \in W$

Thus W contains 0 and the additive inverse of each of its elements

∴(W,+) is an abelian group.

Hence w is an additive subgroup of V.

Also $u \in W$ and $\alpha \in F$

$\Rightarrow \alpha u \in W$.

Since the elements of W are the elements of V the other axioms of a vector space are true Nw.

Hence W is a subspace of V

Theorem:5.3

Let V be a vector space over a field F . Non empty subset W of V is a subspace of V iff $u,v \in F$.

And $\alpha\beta \in F \rightarrow \alpha u + \beta v \in W$.

Proof:

Let W be a subspace of V. Let $u,v \in W$. and $\alpha, \beta \in F$

Then by theorem 5.2 $\beta v \in W$and hence $\alpha u + \beta v \in W$

Conversely let $u,v \in W$ and $\alpha, \beta \in F$

$\Rightarrow \alpha u + \beta v \in W$.

Taking $\alpha = \beta = 1$ we get $u,v \in W$.

$$\Rightarrow u + v \in W$$

Taking $\beta = 0$ we get $\alpha \in F$ and $u \in W \Rightarrow \alpha u \in W$.

∴W is a subspace of r

Example 1:

{0} and V are subspaces of any vector space V. They are called the trivial subspaces of V.

Example2:

$W=\{(a,0,0) / a \in R\}$ is a subspace of R^3, for let $u = (a,0,0)$,

$v = (b,0,0) \in W$ and $\alpha, \beta \in R$

Soln:

Then $\alpha u + \beta v = \alpha (a, 0,0) + \beta (b, 0,0)$

$=(\alpha a, 0,0) + (\beta b, 0,0)$

$=(\alpha a, \beta b, 0,0) \in W$

Hence W is a subspace of R^3

Note :

Geometrically W Consists of all points on the x-axis in the Euclidean 3 space

Example : 3

In $R^3 W=\{ (k_a, k_b, k_c) / k \in R\}$ is a subspace of R^3

Soln :

For if $u = (k_{1a}, k_{1b}, k_{1c})$ $v = (k_{2a}, k_{2b}, k_{2c}) \in W$

$u, v \in W$ and $\alpha, \beta \in R$

then, $\alpha u + \beta v = \alpha (k_{1a}, k_{1b}, k_{1c}) + \beta (k_{2a}, k_{2b}, k_{2c})$

$= (\alpha k_{1a}, \alpha k_{1b}, \alpha k_{1c}) + (\beta k_{2a}, \beta k_{2b}, \beta k_{2c})$

$= (\alpha k_1 + \beta k_2) a, (\alpha k_1 + \beta k_2) b + (\alpha k_1 + \beta k_2) c$

Hence W is a subspace of R^3

Unit – 1

Subspaces

Problem – 1

Prove that the intersection of two subspaces of a vector space is a subspace.

Solution.

Let A and B be two subspace of a vector space V over a field F.

We claim that $A \cap B$ is a subspace of V.

Clearly $0 \in A \cap B$ and hence $A \cap B$ is non – empty.

Now ,let $u, v \in A \cap B$ and $\alpha, \beta \in F$.

Then $u, v \in A$ and $u, v \in B$.

$\alpha u + \beta v \in A$ and $\alpha u + \beta v \in B$.

(Since A and B are subspaces)

$\therefore \alpha u + \beta v \in A \cap B$.

Hence $A \cap B$ is a subspace of V.

Problem. 2

Prove that the union of two subspaces of a vector space need not be a subspace.

Solution.

Let $A = \{ (a, 0,0) / a \in R \}$

$B = \{(0,b,0) / b \in R\}$

Clearly A and B are subspace of R^3 (example 2 of 5.2)

However $A \cup B$ is a not a subspace of R^3.

For (1,0,0) and (0,1,0) ∈ A ∪ B.

But (1,0, 0) + (0,1,0) = (1,1,1) ∉ A∪ B.

Problem – 3.

Prove that the union of two subspaces of a vector space is a subspace iff one is contanied in the other.

Solution.

Refer theorem

Let H and K be two subgroups of G such that one is contained in the other. Hence either H ⊆K or K⊆H.

∴H ∪K =K or H∪ K= H. Hence H∪ K is a subgroup of G.

Conversely, Suppose H ∪ K is a subgroup of G. We claim that H⊆ K or K ⊆ H.

Suppose that H is not contained in K and K is not contained in H . Then there exist elements a, b such that

a∈H and a∉ K … … (1)

b∈ K and b ∉ H … (2)

clearly a,b ∈ H ∪ k . Since H ∪ K is a subgroup of G, ab ∈ H ∪ K .Hence ab ∈ H or ab ∈ K.

case (1) Let ab ∈ H. Since a ∈ K, a^{-1}∈ H.

Hence a^{-1} (ab) = b∈ H which is a contradiction to (2).

Case(2)

Let ab ∈ K . Since b ∈ k, b^{-1} ∈ K.

Hence (ab)b^{-1} = a ∈ K Which is a contradiction to (1).

Hence Our assumption that H is not contained in K and K is not contained in H is false .

H⊆ K or K ⊆ H.

Problem:

If A and B are subspaces of V prove that $A+B=\{v\epsilon V \backslash v=a+b, b\epsilon B\}$ is a subspace of V. Furthere show that A+B is the smellest subspace containing A and B. (i.e) If W is any any subspace of V containing A and B then W contains A+B.

Soln:

Let $v_1,v_2\epsilon A+B$ and $\alpha\epsilon F$.
Then $v_1=a_1+b_1$, $v_2=a_2+b_2$ where $a_1,a_2\epsilon A$ and $b_1,b\epsilon B$.
Now, $v_1+v_2=(a_1+b_1)+(a_2+b_2)$

$$=(a_1+a_2)+(b_1+b_2)\epsilon A+B.$$

Also, $\alpha(a_1+b_1)=\alpha a_1+\alpha b_1\epsilon A+B$

Hence A+B is a subspace of V. Clearly $A\subseteq A+B$ and $B\subseteq A+B$.Ş

Now, let we be any subspace of V containing A and B.

To Prove: $A+B\subseteq W$.

Let $v\epsilon A+B$. Then $v=a+b$ where $a\epsilon A$ and $b\epsilon B$.

$a+b=v\epsilon W$.

$A+B\subseteq W$ so that A+B is the smallest subspace of V containing A and B.

Problem: 5

Let A and B be subspace of a vecor space V. Then $A\cap B=\{0\}$ iff every vector $v\epsilon A+B$ can be uniquely expressed in the form $v=a+b$ where $a\epsilon A$ and $b\epsilon B$.

Soln:

Let $A\cap B=\{0\}$. Let $v\epsilon A+B$.

Let $v=a_1+b_1=a_2+b_2$ where $a_1,a_2\epsilon A$ and $b_1,b_2\epsilon B$.

Then $a_1-a_2= b_1-b_2$.

But $a_1-a_2\epsilon A$ and $b_2-b_1\epsilon B$.

Hence $a_1-a_2, b_2-b_1\epsilon A\cap B$.

Since $A\cap B=\{0\}$, $a_1-a_2=0$ and $b_2-b_1=0$ so that $a_1=a_2$ and $b_1=b_2$. Hence the expression of v in the form a+b where $a\epsilon A$ and $b\epsilon B$ is unique.

form a+b where $a\epsilon A$ and $b\epsilon B$.

We claim that $A\cap B=\{0\}$.

If $A\cap B\neq\{0\}$, let $v\epsilon A\cap B$ and $v\neq 0$.

Then $0=v-v=0+0$. Thus 0 has expressed in the form a+b in two different ways which is a contradiction. Hence $A\cap B=\{0\}$.

DEFINITION:

Let A and B be subspaces of a vector space V. Then V is called the **direct sum** of A and B if **(i)** A+B=V **(ii)** $A\cap B=\{0\}$.

If Vis the direct sum of A and B we write V=A direct sum of B.

Note:

V= A direct sum of B iff every element of V can be uniquely expressed in the form a+b where $a\epsilon A$ and $b\epsilon B$.

EXAMPLE: 1

In $v_3(R)$ Let A={(a,b,0)\a,b∈R} B={(0,0,c)\c∈R}

Clearly A and B are subspace of V and A∩B={0}.

V= (a,b,c)∈ $V_3(R)$

Then V=(a,b,0)+(0,0,c)

So that A+B= $V_3(R)$

Hence $V_3(R)$= A direct sum of B.

EXAMPLE: 2

In $M_3(R)$, let A be the set of all matrices of the form A=$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ B=$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

Let u= $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ V= $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

$$\alpha u+\beta v= \alpha \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

$$=\begin{pmatrix} \alpha a & \alpha b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta c & \beta d \end{pmatrix} \qquad \alpha,\beta\in M_2(R)$$

$$= \begin{pmatrix} \alpha a & \alpha b \\ \beta c & \beta d \end{pmatrix}$$

Clearly A and B are subspace of $M_2(R)$.

$$A+B= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M_2(R)$$

$A\cap B= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $M_2(R)$ = A direct sum of B.

Theorm 5.4. Let V b a vector space overF and W Subspace of v.Let V/W= {W+v/v € V). Then V/W is a vector space over F under the following operations.

1) (W+v1) + (W + v2) = W+ v1 + v2.
2) a(W + v1)= W+ av1

Proof.

Since W is a subspace of V it is a subgroup of (V, +).

Since (V, +) is abelian, W is a normal subgroup of V, +) so that (i) is a well defind operation.

Now we shall prove that (ii) is a well defind operation.

W +v1= W+v2
V1 -v2 €W (since W is a supspace)
α(v1-v2) €W
αv1€W+αv2
W+αv1=W+αv2
Hence (2) is well defined operation
Now, let

W is a group under +.

Further (W + v1) + (W+

v2) = W+v1+v2=W+

v2+v1

(W+ v2) + (W+ v2)

Hence V/W is an abelian group.

Now, let α, β€ F
α(W +v1)+(W+ v2))=α(W +v1 +v2)

'= W+α(v1 + v2)

=W+ αv1 +αv2

= (W+αv1) + (W+αv2)

=α (W +v1) + α (W + v2)

(α+β)(W+ v1) = W + (α + β)v1

= W+αv1 +βv1

= (W+ αv1)+(W + βv1)

=α(W+v1)+β(W+v1)

α[β(W+v1)]= α(W+βv1)

=(αβ)(W+v1)

1[W+v1]=W+1v1

=W+v1

Hence V/W is a vector space .

The vector space V/W is called the
quotient space of V andW

LINEAR TRANSFORMATION

Definition:

Let V and W be vector spaces over a field F. A mapping $T : V \to W$ is called a homomorphism if

(a) $T(u + v) = T(u)+T(v)$ and
(b) $T(\alpha u) = \alpha T(u)$ where $\alpha \in F$ and $u, v \in V$.

A homomorphism T of vector spaces is also called a linear transformation.

(I) If T is 1-1 then T is called monomorphism.
(II) If T is onto then T is called an epimorphism.
(III) If T is 1-1 and onto T is called isomorphism.
(IV) Two vector spaces V and W are said to be isomorphic if there exists an isomorphism $T : V \to W$ and we write V isomorphism to W.
(V) A linear transformation $T : V \to F$ is called a linear functional.

Examples:

1. Let V be a vector space over a field F and W a subspace of V. Then $T : V \to V/W$ defined by $T(v)=W + v$ is a linear transformation, for,
   $T(v_1+v_2)=W+(v_1+v_2)$
   $=(W + v_1)+(W + v_2)$
   $=T(v_1)+T(v_2)$
   Also $T(\alpha v_1)=W + \alpha v_1$
   $=\alpha(W + v_1)$
   $=\alpha T(v_1)$.
   This is called the natural homomorphism from V to V/W. Clearly T is onto and hence T is an epimorphism.
2. $T : V_3(R) \to V_3(R)$ defined by $T(a, b, c)=(a,0,0)$ is a linear transformation.
3. $T : R^2 \to R^2$ defined by $T(a, b)=(2a-3b, a+4b)$ is a linear transformation.
   Let $u = (a, b)$ and $v = (c, d)$ and $\alpha \in R$.
   Therefore, $T(u + v)=T((a, b) + (c, d))$

$=T(a + c, b + d)$

$=(2(a + c) - 3(b + d), (a + c) + 4(b + d))$

$=(2a + 2c - 3b - 3d, a + c + 4b + 4d)$

$=(2a - 3b + 2c - 3d, a + 4b + c + 4d)$

$=(2a - 3b, a + 4b) + (2c - 3d, c + 4d)$

$=T(a, b)+T(c, d)$

$=T(u)+T(v)$.

Also, $T(\alpha u)=T(\alpha(a, b))$

$=T(\alpha a, \alpha b)$

$=(2\alpha a - 3\alpha b, \alpha a + 4\alpha b)$

$=\alpha(2a - 3b, a + 4b)$

$=\alpha T(a, b)$

$=\alpha T(u)$.

Hence T is a linear transformation.

Theorem: 5.5

Let $T: V \to W$ be a linear transformation .Then $T(V) = \{T(V) / v \in V\}$ is a Subspace of W.

Proof:

Let w_1 and $w_2 \in T(V)$ and $\alpha \in F$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$.

Hence $w_1 + w_2 = T(v_1) + T(v_2)$

$= T(v_1) + T(v_2) \in T(V)$

Similarly $\alpha w_1 = \alpha T(v_1) = T(\alpha v_1) \in T(V)$

Hence T(V) is a subspace of W.

Definition:

Let V and W be a vector spaces over a field F and $T: V \to W$ be a linear Transformation. Then the kernel of T is defined to be $\{v / v \in V$ and $T(V)=0\}$ and is denoted by ker T.

Thus $\ker T = \{v / v \in V \text{ and } T(v) = 0\}$

For example, in example 1, $\ker T = V$.

In example 2, $\ker T = \{0\}$

In example 5, ker T is the set of all constant polynomials.

Note:

Let $T : V \to W$ be a linear transformation. Then T is a homomorphism iff $\ker T = \{0\}$

Theorem 5.6 : (Fundamental theorem of homomorphism)

Let V and W be vector spaces over a field F and T : V→ W be an epimorphism.

Then (i) ker T = V_1 is a subspaces of V and

(ii) $\frac{V}{V_1} \cong W$

Proof:

(i) Given V_1= ker T

$= \{v/v \in V \text{ and } T(v)=0\}$

Clearly T(0) = 0. Hence $0 \in$ ker T = V_1

∴V1 is nonempty subset of V.

Let u, v ∈ ker T and α, β∈F

∴T(u) = 0and T(v) = 0

Now $T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha T(u) + \beta T(v)$

$= \alpha 0 + \beta 0$

$= 0$

$\therefore 0\alpha u + \beta v \in \ker T$

ker T is a subspace of V.

(ii) We define a map $\varphi : \frac{V}{V_1} \to W$ by φ(V1+v)=T(v)

φ is well defined.

Let $V_1+v=V_1+w$

$v \in V_1+w$

$v = v_1+w$ where $v_1 \in V_1$

$T(v) = T(v_1+ w)=T(v_1) + T(w)$

$0 + T(w) = T(w)$

$\varphi (V_1+ v) = \varphi(V_1+ w)$

φ is 1-1

$\varphi (V_1+v)=\varphi(V_1+w)$

$\Rightarrow T(v)=T(w)$

$\Rightarrow T(v)-T(w)=0$

$\Rightarrow T(v)+T(-w)=0$

$\Rightarrow T(v-w)=0$

$\Rightarrow v-w \in \ker T=V_1$

$\Rightarrow v \in V_1+w$

$\Rightarrow V_1+v=V_1+w$

φ is onto

Let $w \in W$.

Since T is onto there exists $v \in V$ such that $T(v)=w$.

$\therefore \varphi(V_1+v)=w$

φ Is a homomorphism.

$$\begin{aligned}\varphi[(V_1+v)+(V_1+w)] &= \varphi (V_1+(v+w)] \\ &= T(v+w) \\ &= T(v)+T(w) \\ &= \varphi(V_1+v)+ \varphi(V_1+w)\end{aligned}$$

THEOREM 5.7

Let V be a vector space over a field F. Let A and B be subspaces of V.

Then $A+B/A \cong B/A\cap B$

Proof :

We know that A+B is a subspace of V containing A.

Hence A+B/A Is also a vector space over F.

An element of A+B/A is of the form A+ (a+b) where $a\epsilon A$ and $b\epsilon B$.

But A+a =A.

Hence an element of A+B/B is of the form A+b.

Now, consider $f: B\rightarrow A+B/A$ defined by

F(b) = A+b.

Clearly f is onto.

Also $f(b_1+b_2) = A+(b_1+b_2)$

$= (A+b_1) + (A+b_2)$

$= f(b_1) +f(b_2)$

And $f(\alpha b_1) = A+\alpha b_1$

$=\alpha(A+b_1)$

$=\alpha\ f(b_1)$

Hence f is a linear transformation.

Let k be the kernel of f.

Then $K = \{b/b \epsilon B, A+b, A+b = A\}$.

Now , A+b = A iff $b \in A$.

Hence $K = A \cap B$.

$B/A \cap B \cong A+B/A$.

THEOREM 5.8

Let V and W be vector spaces over a field F .let L (V,W) represent the set of all linear transformation from V to W. Then L(V,W) itself is a vector space over F under addition and scalar multiplication

Defined by $(f+g)(v) = f(v) + g(v)$ and $(\alpha f)(v) = \alpha f(v)$.

Proof:

Let f, $g \in L(V,W)$ and $v_1, v_2 \in V$

Then $(f+g)(v_1 + v_2) = f(v_1+v_2) + g(v_1+v_2)$

$= f(v_1) + f(v_2) + g(v_1) + g(v_2)$

$= f(v_1) + g(v_1) + f(v_2) + g(v_2)$

$= (f+g)(v_1) + (f+g)(v_2)$

Also $(f+g)(\alpha v) = f(\alpha v) + g(\alpha v)$

$= \alpha f(v) + \alpha g(v)$

$= \alpha[f(V) + g(v)]$

$= \alpha(f+g)(v)$.

Hence $(f+g) \in L(V,W)$

Now, $(\alpha f)(v_1+v_2) = (\alpha f) + (v_1) + (\alpha f)(v_2)$

$= \alpha[f(v_1) + f(v_2)]$

$= \alpha f(v_1+v_2)$

Also $(\alpha f)(\beta v) = \alpha[f(\beta v)] = \alpha[\beta f(v)]$

Hence $\alpha f \in L(V,W)$

Addition defined on L(V,W) is obviously commutative and associative .

The function f : V→W defined by f(v) =**0** for all $v\epsilon V$ *is clearly* a linear transformation and is the additive identity of L(V,W).

Further (-f) : V→W defined by

(-f) (v) = -f(v) is the additive inverse of f.

Thus L(V,W) is an abelian group under addition.

The remaining axioms for a vector space can be easily verified,

Hence L(V,W) is a vector space over F.